Untying the Knot between a Stochastic Program and its Distribution

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Evidence that Managing an Investment Portfolio is Difficult

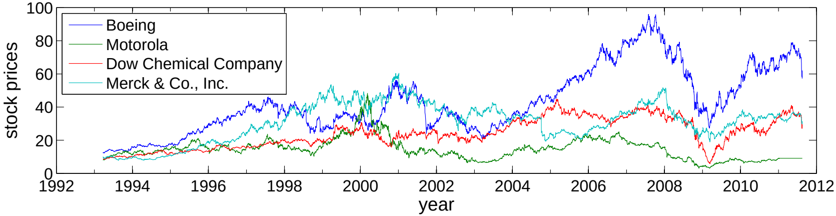
Value on Jan 1st 2009 of each dollar contribution made to the Caisse de Dépôts et de Placements

Date of contribution	CDPQ	1-year guaranteed certificates
Jan 1st, 2008	\$0,75	\$1,03
Jan 1st, 2007	\$0,79	\$1,05
Jan 1st, 2006	\$0,91	\$1,07
Jan 1st, 2005	\$1,04	\$1,09
Jan 1st, 2004	\$1,17	\$1,10
Jan 1st, 2003	\$1,35	\$1,12
Jan 1st, 2002	\$1,22	\$1,13
Jan 1st, 2001	\$1,16	\$1,18
Jan 1st, 2000	\$1,23	\$1,23

Why are Financial Investments so Fragile?

Some reasons:

- A wide range of financial securities can be used for investment
- Securities have become very complex
- The risks involved are difficult to evaluate
- Limited knowledge of how the market will behave in the future



Are Airlines Adventurous in their Fleet Acquisition?

- Fleet composition is a difficult decision problem:
  - Fleet contracts are signed 10 to 20 years ahead of schedule.
  - Many factors are still unknown at that time:
    e.g., passenger demand, fuel prices, etc.
- Yet, most airline companies sign these contracts based on a single scenario of what the future may be.
- Are airlines companies at risk of going bankrupt?

Stochastic Programming Approach

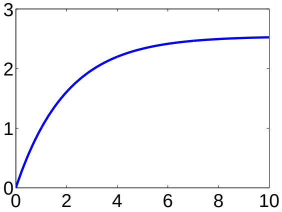
Let's consider the stochastic programming problem:

$$\underset{\mathbf{x}\in\mathcal{X}}{\text{maximize}} \quad \mathbb{E}\left[u(\mathrm{h}(\mathbf{x},\boldsymbol{\xi}))\right]$$

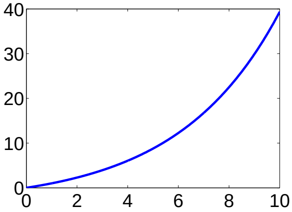
where $\mathbf{x}$ = decisions and $\boldsymbol{\xi}$ = uncertain parameters.

Here, we assume that we know:

- The distribution of the random vector $\boldsymbol{\xi}$
- A utility function that matches investor's attitude to risk



Retirement Fund



Vegas Budget

Difficulty of developing a probabilistic model

Developing an accurate probabilistic model requires heavy engineering efforts:

- Need to collect enough observations
- Need to consult with experts of the field of practice
- Need to make simplifying assumptions

Yet, there are inherent pitfalls in the process:

- Expecting that a scenario might occur does not determine its probability of occurring
- Unexpected event (e.g., economic crisis) might occur
- The future might actually not behave like the past

Limits of Expected Utility: Ellsberg Paradox

Consider an urn with 30 blue balls and 60 other balls that are either red or yellow (you don't know how many are red or yellow).

Experiment 1: Choose among the following two gambles

- Gamble A: If you draw a blue ball, then you win 100$
- Gamble B: If you draw a red ball, then you win 100$

Experiment 2: Choose among the following two gambles

- Gamble C: If you draw blue or yellow ball, then you win 100$
- Gamble D: If you draw red or yellow ball, then you win 100$

If you clearly prefer Gamble A & D, then you cannot be thinking in terms of expected utility.

Untying the SP from a Specific Distribution

- Let's consider that the choice of F is ambiguous
- Use available information to define $\mathcal{D}$, such that $F \in \mathcal{D}$
- We are faced with a multi-objective optimization problem:

$$\underset{\mathbf{x}\in\mathcal{X}}{\text{maximize}} \quad \{\, \mathbb{E}_F[u(\mathsf{h}(\mathbf{x},\boldsymbol{\xi}))] \,\}_{F\in\mathcal{D}}$$

- Distributionally Robust Optimization values the lowest performing one

$$(DRSP) \qquad \underset{\mathbf{x}\in\mathcal{X}}{\text{maximize}} \; \min_{F\in\mathcal{D}} \mathbb{E}_F[u(\mathsf{h}(\mathbf{x},\boldsymbol{\xi}))]$$

  - Introduced by H. Scarf in 1958
  - Recently, we found ways of solving some DRSP's efficiently [Popescu (2007), Bertsimas et al., Natarajan et al., Delage et al. (2010)]
  - Possible to promote performance differently depending on F [Föllmer et al. (2002), Li et al. (2011)]

Outline

1. Introduction
2. Distributionally Robust Optimization
3. Distributions Can Be Misleading
4. Value of Stochastic Modeling
5. Conclusion

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Assumptions on Objective Function

Let's make two assumptions about $\mathbb{E}[u(\mathrm{h}(\mathbf{x},\boldsymbol{\xi}))]$.

1. The utility function is piecewise linear concave :

$$u(y) = \min_{1\leq k\leq K} \quad a_k y + b_k \quad ,$$

2. The profit function is the maximum of a linear program with uncertainty limited to objective

$$\begin{aligned} \mathrm{h}(\mathbf{x},\boldsymbol{\xi}) \;:=\; \max_{\mathbf{y}}. \quad & \mathbf{c}_1^{\mathsf{T}}\mathbf{x} + \boldsymbol{\xi}^{\mathsf{T}} C_2 \mathbf{y} \\ \text{s.t.} \quad & A\mathbf{x} + B\mathbf{y} \leq \mathbf{b} \end{aligned}$$

Resolving Distributional Set from Data

- Question:
  - We have in hand i.i.d. samples $\{\boldsymbol{\xi}_i\}_{i=1}^M$
  - We know that $\mathbb{P}(\boldsymbol{\xi} \in \mathcal{S}) = 1$ and $\mathcal{S} \subseteq \mathcal{B}(\mathbf{0}, R)$
  - We can estimate the mean and covariance matrix:

$$\hat{\boldsymbol{\mu}} = \frac{1}{M}\sum_{i=1}^M \boldsymbol{\xi}_i \qquad \hat{\Sigma} = \frac{1}{M}\sum_{i=1}^M (\boldsymbol{\xi}_i - \hat{\boldsymbol{\mu}})(\boldsymbol{\xi}_i - \hat{\boldsymbol{\mu}})^\mathsf{T}$$

  - What do we know about the distribution behind these samples?

- Answer:

$$\mathcal{D}(\gamma) = \left\{ F \left| \begin{array}{l} \mathbb{P}(\boldsymbol{\xi} \in \mathcal{S}) = 1 \\ \|\mathbb{E}\left[\boldsymbol{\xi}\right] - \hat{\mu}\|^2_{\hat{\Sigma}^{-1/2}} \leq \gamma_1 \\ \mathbb{E}\left[(\boldsymbol{\xi} - \hat{\boldsymbol{\mu}})(\boldsymbol{\xi} - \hat{\boldsymbol{\mu}})^\mathsf{T}\right] \preceq (1+\gamma_2)\hat{\Sigma} \end{array} \right. \right\}$$

  - With prob. $> 1 - \delta$ the distribution is contained in $\mathcal{D}(\gamma)$ for some $\gamma_1 = O\left(\frac{R^2}{M}\log(1/\delta)\right)$ and $\gamma_2 = O\left(\frac{R^2}{\sqrt{M}}\sqrt{\log(1/\delta)}\right)$.

The DRSP is a SDP

- The DRSP problem with $\mathcal{D}(\gamma)$ is equivalent to

$$\begin{array}{lll} \max\limits_{\mathbf{x},\,\mathbf{Q},\mathbf{q},r} & r - \left(\gamma_2\hat{\Sigma} + \hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}^{\mathsf{T}}\right) \bullet \mathbf{Q} - \hat{\boldsymbol{\mu}}^{\mathsf{T}}\mathbf{q} - \sqrt{\gamma_1}\,\|\hat{\Sigma}^{1/2}(\mathbf{q} + 2\mathbf{Q}\hat{\boldsymbol{\mu}})\| \\ \text{s.t.} & r \leq \min\limits_{\boldsymbol{\xi}\in\mathcal{S}} u(\mathrm{h}(\mathbf{x},\boldsymbol{\xi})) + \boldsymbol{\xi}^{\mathsf{T}}\mathbf{q} + \boldsymbol{\xi}^{\mathsf{T}}\mathbf{Q}\boldsymbol{\xi} \quad (\star) \\ & \mathbf{Q} \succeq 0 \end{array}$$

- If $\mathcal{S}$ = polygon or ellipsoid, then DRSP equivalent to semi-definite program.
  E.g., when $\mathcal{S} = \mathbb{R}^m$, Constraint $(\star)$ can be replaced by

$$\begin{bmatrix} \mathbf{Q} & (\mathbf{q} + a_k C_2 y_k)/2 \\ (\mathbf{q} + a_k C_2 y_k)^{\mathsf{T}}/2 & a_k c_1^{\mathsf{T}}\mathbf{x} + b_k - r \end{bmatrix} \succeq 0\ ,\ \forall k$$

The Robustness of the Deterministic Solution

If we are risk neutral we might not even need distribution information

Theorem

The solution of

$$\underset{\mathbf{x}\in\mathcal{X}}{\text{maximize}} \quad \mathbb{E}[h(\mathbf{x},\mu)]$$

is optimal with respect to

$$\underset{\mathbf{x}\in\mathcal{X}}{\text{maximize}} \quad \inf_{F\in\mathcal{D}(\mu,\Psi)} \quad \mathbb{E}_F[h(\mathbf{x},\xi)] \ ,$$

for any set of convex functions Ψ *with*

$$\mathcal{D}(\mu,\Psi) = \left\{ F \,\middle|\, \begin{array}{l} \mathbb{E}[\xi] = \mu \\ \mathbb{E}[\psi(\xi)] \leq 0 \ , \ \forall\, \psi \in \Psi \end{array} \right\} .$$

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Distributionally Robust Portfolio Optimization

Let's consider the case of portfolio optimization:

$$\max_{\mathbf{x}\in\mathcal{X}} . \quad \min_{F\in\mathcal{D}} \mathbb{E}_F[u(\boldsymbol{\xi}^\mathsf{T}\mathbf{x})] \quad ,$$

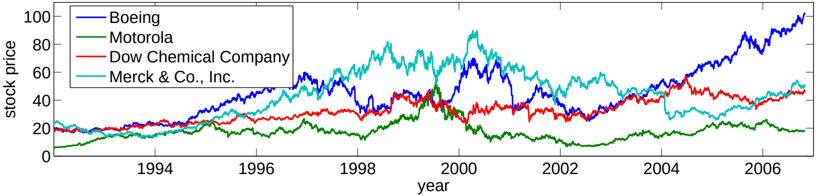
where x_i is how much is invested in stock i with future return ξ_i.

Does the robust solution perform better than a stochastic programming solution?

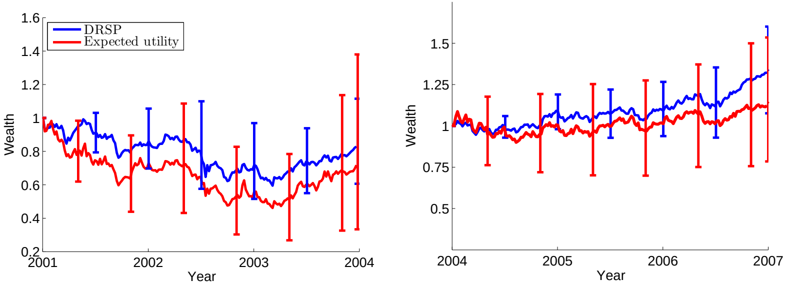
$$\mathcal{D} = \mathcal{D}(\gamma) \qquad \text{vs.} \qquad \mathcal{D} = \{\hat{F}\}$$

Experiments in Portfolio Optimization

30 stocks tracked over years 1992-2007 using Yahoo! Finance



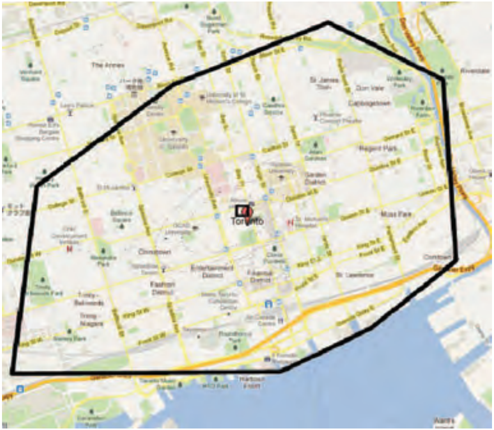
Wealth Evolution for 300 Experiments



- 10% and 90% percentiles are indicated periodically.
- 79% of time, the DRSP outperformed the exp. utility model
- 67% improvement on average using DRSP with $\mathcal{D}(\gamma)$

Multi-Vehicle Routing on a Planar Region

- Divide a planar region into K subregions, each serviced by a different vehicle, so that the total workload be most evenly distributed among the fleet



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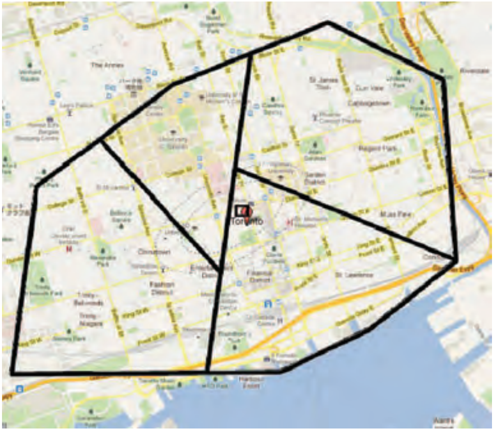

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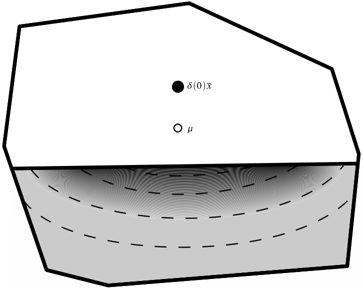
Distributionally Robust Partitioning

- Given $\mathcal{D}$, we partition so that the largest workload over the worst distribution of demand points is as small as possible

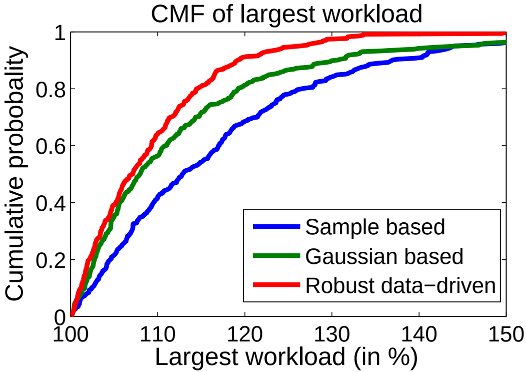
$$\min_{\{\mathcal{R}_1,\mathcal{R}_2,\ldots,\mathcal{R}_K\}} \sup_{F\in\mathcal{D}} \left\{ \max_i \mathbb{E}[TSP(\{\xi_1,\xi_2,\ldots,\xi_N\}\cap\mathcal{R}_i)] \right\} \quad ,$$

- A side product is to characterize for any partition what is a worst-case distribution of demand locations



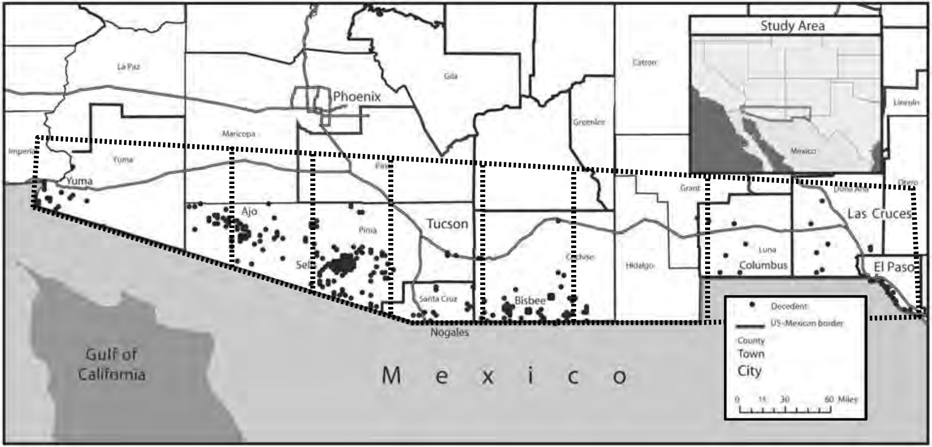
Distributionally Robust Partitioning

We simulated three partition schemes on a set of randomly generated parcel delivery problems where the territory needed to be divided into two regions and the demand is drawn from a mixture of truncated Gaussian distribution



Border Patrol Workload Partitioning

Robust partitions of the USA-Mexico border obtained using our branch & bound algorithm.



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The Value of Stochastic Modeling

Consider the situation:

1. We know of a set $\mathcal{D}$ such that $F \in \mathcal{D}$
2. We have a candidate solution $\mathbf{x}_1$ in mind
3. Is it worth developing a stochastic model: $\mathcal{D} \rightarrow F$?
   - (a) If yes, then develop a model & solve it
   - (b) Otherwise, implement $\mathbf{x}_1$

The Value of Stochastic Modeling ($\mathcal{VSM}$) gives an optimistic estimate of the value of obtaining perfect information about F.

$$\mathcal{VSM}(\mathbf{x}_1) \;:=\; \sup_{F \in \mathcal{D}} \left\{ \max_{\mathbf{x}_2} \mathbb{E}_F[\mathrm{h}(\mathbf{x}_2, \boldsymbol{\xi})] - \mathbb{E}_F[\mathrm{h}(\mathbf{x}_1, \boldsymbol{\xi})] \right\}$$

Theorem

Unfortunately, evaluating $\mathcal{VSM}(\mathbf{x}_1)$ exactly is NP-hard in general.

Bounding the Value of Stochastic Modeling

Theorem

If $\mathcal{S} \subseteq \{\boldsymbol{\xi} \,|\, \|\boldsymbol{\xi}\|_1 \leq \rho\}$*, an upper bound can be evaluated in* $O(d^{3.5} + d\, T_{DCP})$ *using:*

$$\begin{aligned} \mathcal{UB}(\mathbf{x}_1, \bar{\mathbf{y}}_1) \;:=\; \min_{s,\mathbf{q}} \quad & s + \boldsymbol{\mu}^\mathsf{T}\mathbf{q} \\ \text{s.t.} \quad & s \geq \alpha(\rho\mathbf{e}_i) - \rho\mathbf{e}_i^\mathsf{T}\mathbf{q} \;, \forall\, i \in \{1, ..., d\} \\ & s \geq \alpha(-\rho\mathbf{e}_i) + \rho\mathbf{e}_i^\mathsf{T}\mathbf{q} \;, \forall\, i \in \{1, ..., d\}\,, \end{aligned}$$

where $\alpha(\boldsymbol{\xi}) = \max_{\mathbf{x}_2} h(\mathbf{x}_2, \boldsymbol{\xi}) - h(\mathbf{x}_1, \boldsymbol{\xi}; \bar{\mathbf{y}}_1)$.

- $\mathcal{UB}$ only uses information about $\boldsymbol{\mu}$ and $\mathcal{S}$
- $\mathcal{UB}$ simplifies the structure of $\mathcal{S}$
- $\mathcal{UB}$ assumes the candidate decision $\mathbf{y}_1$ cannot adapt to $\boldsymbol{\xi}$

Mathematical formulation for Fleet Mix Problem

The fleet composition problem is a stochastic mixed integer LP

Fleet mix

$$\underset{\mathbf{x}}{\max.}\quad \mathbb{E}\,[-\underbrace{\mathbf{o}^{\mathsf{T}}\mathbf{x}}_{\text{ownership cost}} + \underbrace{h(\mathbf{x},\tilde{\mathbf{p}},\tilde{\mathbf{c}},\tilde{\mathbf{L}})}_{\text{future profits}}\,]\ ,$$

with $h(\mathbf{x},\tilde{\mathbf{p}},\tilde{\mathbf{c}},\tilde{\mathbf{L}}) :=$

$$\max_{z\geq 0,y\geq 0,w} \quad \sum_k \Big(\sum_i \overbrace{\tilde{p}_i^k w_i^k}^{\text{flight profit}} - \overbrace{\tilde{c}_k(z_k - x_k)^+}^{\text{rental cost}} + \overbrace{\tilde{L}_k(x_k - z_k)^+}^{\text{lease revenue}} \Big)$$

$$\text{s.t.} \quad \left. w_i^k \in \{0,1\}\ ,\ \forall\, k, \forall\, i \quad \& \quad \sum_k w_i^k = 1\ ,\ \forall\, i \right\}\ \text{Cover}$$

$$\left. y^k_{g\in\text{in}(v)} + \sum_{i\in\text{arr}(v)} w_i^k = y^k_{g\in\text{out}(v)} + \sum_{i\in\text{dep}(v)} w_i^k\ ,\ \forall\, k, \forall\, v \right\}\ \text{Balance}$$

$$\left. z_k = \sum_{v\in\{v|\text{time}(v)=0\}} \big(y^k_{g\in\text{in}(v)} + \sum_{i\in\text{arr}(v)} w_i^k\big)\ ,\quad \forall k \right\}\ \text{Count}$$

Experiments in Fleet Mix Optimization

We experimented with three test cases :

1. 3 types of aircrafts, 84 flights, $\sigma_{\tilde{p}_i}/\mu_{\tilde{p}_i} \in [4\%, 53\%]$
2. 4 types of aircrafts, 240 flights, $\sigma_{\tilde{p}_i}/\mu_{\tilde{p}_i} \in [2\%, 20\%]$
3. 13 types of aircrafts, 535 flights, $\sigma_{\tilde{p}_i}/\mu_{\tilde{p}_i} \in [2\%, 58\%]$

Results:

Test cases	Computation Times			Upper bound for VSM
	DCP	SP (100 scen.)	$\mathcal{UB}$	
#1	0.6 s	3 min	12 sec	6%
#2	1 s	14 min	40 sec	1%
#3	5 s	21 h	2 min	7%

Conclusions:

- It's wasteful to invest more than 7% of profits in extra info

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Conclusion & Future Work

- Many forms of the DRSP are tractable
- Some actually reduce to the DCP
- Thinking we know the distribution can be misleading
- Knowing the actual distribution might not help that much
- There are tools that help estimate how much the true distribution is worth

- Open questions :
  - Can tractable DRSP be made consistent ?
  - Can DRSP be extended to multi-objective problems ?
  - How to deal with ambiguity about one's utility function ?

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Questions & Comments ...

... Thank you!