On a Class of Uncertainty Sets for Distributionally Robust Optimization

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Workshop on Robust Optimization
May 2011

Outline

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Stochastic Programming Approach

Let's consider a stochastic programming problem:

$$\underset{\mathbf{x}\in\mathcal{X}}{\text{maximize}} \quad \mathbb{E}\left[u(\mathsf{h}(\mathbf{x},\boldsymbol{\xi}))\right]$$

where $\mathbf{x}$ = decisions and $\boldsymbol{\xi}$ = uncertain parameters.

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Here, we assume :

- We know the distribution of the random vector $\boldsymbol{\xi}$
- The profit function $\mathrm{h}(\mathbf{x},\boldsymbol{\xi})$ is concave in $\mathbf{x}$ and convex in $\boldsymbol{\xi}$
- We have a piecewise linear utility function that characterizes investor's risk aversion

$$u(y) = \min_{k\in\{1,2,\dots,K\}} \alpha_k y + \beta_k$$

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Yet, there are inherent pitfalls in the process:

- Expecting that a scenario might occur does not determine its probability of occurring
- Unexpected event (e.g., economic crisis) might occur
- The future might actually not behave like the past

Limits of Expected Utility: Ellsberg Paradox

Consider an urn with 30 blue balls and 60 other balls that are either red or yellow (you don't know how many are red or yellow).

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Experiment 2: Choose among the following two gambles

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- Gamble D: If you draw red or yellow ball, then you win 100$

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- Gamble D: If you draw red or yellow ball, then you win 100$

If you clearly prefer Gamble A & D, then you cannot be thinking in terms of expected utility.

The Distributionally Robust Formulation

- We consider that the choice of F is ambiguous
- Use available information to define $\mathcal{D}$, such that $F \in \mathcal{D}$
- Distributionally Robust Optimization values a decision using the lowest performing distribution

$$(DRSP) \qquad \underset{\mathbf{x}\in\mathcal{X}}{\text{maximize}} \; \inf_{F\in\mathcal{D}} \; \mathbb{E}_F[u(\mathsf{h}(\mathbf{x},\boldsymbol{\xi}))]$$

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  - Introduced by H. Scarf in 1958
  - Recently, we found ways of solving some DRSP's efficiently [Popescu (2007), Bertsimas et al., Natarajan et al., Delage et al. (2010)]
  - Growing interest for promoting performance differently depending on F [Föllmer et al. (2002), Li et al. (2011)]

Outline

Definition of $\mathcal{D}(\psi)$

Let $\mathcal{D}(\psi)$ be a set of the form

$$\mathcal{D}(\psi) = \left\{ F \in \mathcal{M} \,\middle|\, \begin{array}{l} \mathbb{P}_F(\boldsymbol{\xi} \in \mathcal{S}) = 1 \\ \mathbf{z}^{\mathsf{T}} \left(\mathbb{E}_F[\psi(\boldsymbol{\xi})] - \mathbf{b} \right) \leq 0 \ , \ \forall \mathbf{z} \in \mathcal{K} \end{array} \right\}$$

such that:

1. The set $\mathcal{S}$ is closed, convex, and bounded
2. The set $\mathcal{K} \subseteq \mathbb{R}^p$ is a convex cone
3. $\forall \mathbf{z} \in \mathcal{K}$, the function $g(\mathbf{z}, \boldsymbol{\xi}) := \mathbf{z}^{\mathsf{T}} \psi(\boldsymbol{\xi})$ is convex in $\boldsymbol{\xi}$
4. Both $\mathcal{S}$ and $\mathcal{K}$ have "nice" representations

$$\mathcal{S} = \{\boldsymbol{\xi} \in \mathbb{R}^m | g_i^S(\boldsymbol{\xi}) \leq 0 \, \forall \, i\}$$

$$\mathcal{K} = \{\mathbf{z} \in \mathbb{R}^p | g_j^K(\mathbf{z}) \leq 0 \, \forall \, j\}$$

$\mathcal{D}(\psi)$ imposes that F be Concentrated

Lemma

Given any random vector ζ such that $F_\zeta \in \mathcal{D}(\psi)$, for all $0 \leq \theta \leq 1$ the distribution of $\zeta' := \theta(\zeta - \mathbb{E}[\zeta]) + \mathbb{E}[\zeta]$ also has a distribution in $\mathcal{D}(\psi)$.

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Proof: (let $\boldsymbol{\mu} := \mathbb{E}[\zeta]$)

1. Since $\mathcal{S}$ is convex and $\mathbb{P}(\zeta \in \mathcal{S}) = 1$, we have that $\mathbb{E}[\zeta] \in \mathcal{S}$, hence
$$\mathbb{P}(\zeta' \in \mathcal{S}) = \mathbb{P}(\theta\boldsymbol{\zeta} + (1-\theta)\boldsymbol{\mu} \in \mathcal{S}) = 1$$
2. By Jensen's inequality for any $\mathbf{z} \in \mathcal{K}$, we have that
$$\begin{aligned}
\mathbb{E}[\mathbf{z}^T\psi(\theta\boldsymbol{\zeta} + (1-\theta)\boldsymbol{\mu})] &\leq \mathbb{E}[\theta\mathbf{z}^T\psi(\zeta) + (1-\theta)\mathbf{z}^T\psi(\boldsymbol{\mu})] \\
&= \theta\mathbb{E}[\mathbf{z}^T\psi(\zeta)] + (1-\theta)\mathbf{z}^T\psi(\boldsymbol{\mu}) \\
&\leq \mathbb{E}[\mathbf{z}^T\psi(\zeta)] \leq \mathbf{z}^T\mathbf{b}
\end{aligned}$$

Examples

- Upper bounds on partial moments : $\mathbb{E}[\max(0, \boldsymbol{\xi}_i - \hat{\mu}_i)^2] \leq b_i$ since equivalent to

$$z(\mathbb{E}[\max(0, \boldsymbol{\xi}_i - \hat{\mu}_i)^2] - b_i) \leq 0, \ \forall \, z \geq 0$$

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- Confidence ellipsoid for mean :

$$(\hat{\boldsymbol{\mu}} - \mathbb{E}[\boldsymbol{\xi}])^T \hat{\Sigma}^{-1} (\hat{\boldsymbol{\mu}} - \mathbb{E}[\boldsymbol{\xi}]) \leq \gamma$$

since equivalent to:

$$\mathbb{E}\left[\mathbf{Z} \bullet \begin{bmatrix} \hat{\Sigma} & (\hat{\boldsymbol{\mu}} - \boldsymbol{\xi}) \\ (\hat{\boldsymbol{\mu}} - \boldsymbol{\xi})^T & \gamma \end{bmatrix}\right] \geq 0, \ \forall \mathbf{Z} \succeq 0$$

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- Upper bound on second order moment matrix

$$\mathbb{E}[(\boldsymbol{\xi} - \hat{\boldsymbol{\mu}})(\boldsymbol{\xi} - \hat{\boldsymbol{\mu}})^T] \preceq \gamma \hat{\Sigma}$$

since equivalent to:

$$\mathbb{E}[\mathbf{Z} \bullet ((\boldsymbol{\xi} - \hat{\boldsymbol{\mu}})(\boldsymbol{\xi} - \hat{\boldsymbol{\mu}})^T - \gamma \hat{\Sigma})] \leq 0, \ \forall \mathbf{Z} \succeq 0$$

Resolving Distributional Set from Data

- Question:
  - We have in hand i.i.d. samples $\{\boldsymbol{\xi}_i\}_{i=1}^M$
  - We know that $\mathbb{P}(\boldsymbol{\xi} \in \mathcal{S}) = 1$ and $\mathcal{S} \subseteq \mathcal{B}(\mathbf{0}, R)$
  - We can estimate the mean and covariance matrix:

$$\hat{\boldsymbol{\mu}} = \frac{1}{M}\sum_{i=1}^{M}\boldsymbol{\xi}_i \qquad \hat{\Sigma} = \frac{1}{M}\sum_{i=1}^{M}(\boldsymbol{\xi}_i - \hat{\boldsymbol{\mu}})(\boldsymbol{\xi}_i - \hat{\boldsymbol{\mu}})^{\mathsf{T}}$$

  - What do we know about the distribution behind these samples?

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  - What do we know about the distribution behind these samples?

- Answer:

$$\mathcal{D}_2(\gamma) = \left\{ F \left| \begin{array}{l} \mathbb{P}(\boldsymbol{\xi} \in \mathcal{S}) = 1 \\ \|\mathbb{E}\left[\boldsymbol{\xi}\right] - \hat{\mu}\|^2_{\hat{\Sigma}^{-1/2}} \leq \gamma_1 \\ \mathbb{E}\left[(\boldsymbol{\xi} - \hat{\boldsymbol{\mu}})(\boldsymbol{\xi} - \hat{\boldsymbol{\mu}})^{\mathsf{T}}\right] \preceq (1 + \gamma_2)\hat{\Sigma} \end{array} \right. \right\}$$

  - With prob. $> 1 - \delta$ the distribution is contained in $\mathcal{D}_2(\gamma)$ for some $\gamma_1 = O\left(\frac{R^2}{M}\log(1/\delta)\right)$ and $\gamma_2 = O\left(\frac{R^2}{\sqrt{M}}\sqrt{\log(1/\delta)}\right)$.

Verifying whether $\mathcal{D}(\psi)$ is Empty or Not

- One finds a distribution that lies in the relative interior of $\mathcal{D}(\psi)$ by verifying that $t^* > 0$ for the problem:

$$\begin{aligned} \max_{F\in\mathcal{M},t\in\mathbb{R}}. \quad & t \\ \text{s.t.} \quad & \mathbf{z}^{\mathsf{T}}(\mathbb{E}_F[\psi(\boldsymbol{\xi})] - \mathbf{b} + t\mathbf{z}_0^*) \leq 0 \,, \; \forall\, \mathbf{z} \in \mathcal{K} \\ & \mathbb{P}_F(\boldsymbol{\xi} \in \mathcal{S}) = 1 \,, \end{aligned}$$

  where $\mathbf{z}_0^*$ is any vector lying in the strict interior of $\mathcal{K}^*$.

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where $\mathbf{z}_0^*$ is any vector lying in the strict interior of $\mathcal{K}^*$.
  - Remember that $\psi(\mathbb{E}_F[\boldsymbol{\xi}]) \leq \mathbb{E}_F[\psi(\boldsymbol{\xi})] \ , \forall\, F \in \mathcal{M}$.

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  where $\mathbf{z}_0^*$ is any vector lying in the strict interior of $\mathcal{K}^*$.
  - Remember that $\psi(\mathbb{E}_F[\boldsymbol{\xi}]) \leq \mathbb{E}_F[\psi(\boldsymbol{\xi})] \, , \forall\, F \in \mathcal{M}$.

- This problem verifies that $\exists\, \boldsymbol{\mu} \in \mathcal{S}$ such that $\mathbf{b} - \psi(\boldsymbol{\mu})$ is in the strict interior of $\mathcal{K}^*$, i.e. a “simple” finite dimensional constraint qualification verification.

Solving the DRSP problem

- Given that $\exists F \in \mathcal{D}(\psi)$ that strictly satisfies all moment constraints, then one can apply duality and solve the equivalent problem:

$$\begin{array}{lll} \underset{\mathbf{x}\in\mathcal{X}}{\text{maximize}} & \underset{F\in\mathcal{M}}{\inf} & \mathbb{E}_F[u(h(\mathbf{x},\boldsymbol{\xi}))] \\ & \text{s.t.} & \mathbb{P}_F(\boldsymbol{\xi}\in\mathcal{S}) = 1 \\ & & \mathbf{z}^\mathsf{T}(\mathbb{E}_F[\psi(\boldsymbol{\xi})] - \mathbf{b}) \leq 0 \;,\; \forall\, \mathbf{z}\in\mathcal{K} \end{array}$$

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$$\begin{aligned} \underset{\mathbf{x}\in\mathcal{X}}{\text{maximize}} \quad \max_{r,\mathbf{z}} \quad & r - \mathbf{b}^{\mathsf{T}}\mathbf{z} \\ \text{s.t.} \quad & u(\mathrm{h}(\mathbf{x},\boldsymbol{\xi})) - r + \mathbf{z}^{\mathsf{T}}\psi(\boldsymbol{\xi}) \ \geq \ 0 \ , \ \forall \boldsymbol{\xi} \in \mathcal{S} \\ & \mathbf{z} \in \mathcal{K} \end{aligned}$$

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\begin{aligned}
\underset{\mathbf{x},\ r,\mathbf{z}}{\text{maximize}} \quad & r - \mathbf{b}^\mathsf{T}\mathbf{z} \\
\text{s.t.} \quad & \alpha_k \mathsf{h}(\mathbf{x},\boldsymbol{\xi}) + \beta_k - r + \mathbf{z}^\mathsf{T}\psi(\boldsymbol{\xi}) \ \geq\ 0\ ,\ \forall\, \boldsymbol{\xi} \in \mathcal{S}\ ,\ \forall\, k \\
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- Applying the robust optimization analysis, we can reformulate each constraints through duality:

$$\min_{\boldsymbol{\xi} \in \mathcal{S}} \;\; \alpha_k \mathrm{h}(\mathbf{x},\boldsymbol{\xi}) + \beta_k - r + \mathbf{z}^{\mathsf{T}}\psi(\boldsymbol{\xi}) \;\geq\; 0$$

Outline

Distributionally Robust Portfolio Optimization

Consider the case of portfolio optimization (with call options):

$$\begin{array}{ll} \max_{\mathbf{x},\mathbf{y}}. & \inf_{F\in\mathcal{D}} \mathbb{E}_F[u(\mathrm{h}(\mathbf{x},\mathbf{y},\boldsymbol{\xi}))] \\ \text{s.t.} & \sum_i x_i + y_i \leq B \\ & \mathbf{x} \geq 0\,, \quad \mathbf{y} \geq 0 \end{array}$$

where x_i = how much is invested in stock i with future return ξ_i, y_i = amount invested in call option purchased on ξ_i, and

$$\begin{array}{ll} \mathrm{h}(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) := \max_{\mathbf{z}}. & \boldsymbol{\xi}^T\mathbf{x} + \sum_i (\xi_i - \bar{\xi}_i) z_i \\ \text{s.t.} & 0 \leq \mathbf{z} \leq \mathbf{y} \end{array}$$

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We bound the distributions with $\mathcal{D}_2(\gamma)$ and consider the support set described by $\mathcal{S} = \{\boldsymbol{\xi} | \mathbf{A}\boldsymbol{\xi} \leq \mathbf{b}\}$

DRSP reduces to SDP

In this context, the robust constraint

$$\min_{\boldsymbol{\xi}\in\mathcal{S}} \;\; \alpha_k \mathrm{h}(\mathbf{x},\boldsymbol{\xi}) + \beta_k + \mathbf{z}^{\mathsf{T}}\psi(\boldsymbol{\xi}) \;\geq\; 0$$

becomes:

$$\underset{\boldsymbol{\xi}\in\mathcal{S}}{\text{minimize}} \;\; \max_{0\leq\mathbf{z}\leq\mathbf{y}} \alpha_k(\boldsymbol{\xi}^T\mathbf{x}+\sum_i(\xi_i-\bar{\xi}_i)z_i)+\beta_k-r+2\mathbf{p}^T\boldsymbol{\xi}+\boldsymbol{\xi}^T\mathbf{Q}\boldsymbol{\xi}-2\hat{\boldsymbol{\mu}}^T\mathbf{Q}\boldsymbol{\xi}$$

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By Sion's minimax and with use of duality we get the problem:

$$\max_{0\leq\mathbf{z}\leq\mathbf{y}}. \max_{\lambda\geq 0} \min_{\boldsymbol{\xi}} \ \ \alpha_k(\boldsymbol{\xi}^T\mathbf{x}+\sum_i(\xi_i-\bar{\xi}_i)z_i)+\beta_k \ \ \ldots \ -2\hat{\boldsymbol{\mu}}^T\mathbf{Q}\boldsymbol{\xi}+\lambda^T(\mathbf{A}\boldsymbol{\xi}-\mathbf{b})$$

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which value is greater or equal to zero iff

$$\begin{bmatrix} \mathbf{Q} & (\alpha_k(\mathbf{x}+\mathbf{z})+2\mathbf{p}-2\mathbf{Q}\hat{\boldsymbol{\mu}}+\mathbf{A}^T\lambda)/2 \\ \ldots & -\alpha_k\bar{\boldsymbol{\xi}}^T\mathbf{z}+\beta_k-\lambda^T\mathbf{b}-r \end{bmatrix} \succeq 0$$

for some $0\leq\mathbf{z}\leq\mathbf{y}$ and $\lambda\geq 0$.

DRSP reduces to SDP II

When the dust settles down we get:

$$\max_{\mathbf{x},\mathbf{y}\; r,\mathbf{Q},\mathbf{P},\mathbf{p},s,\; \mathbf{z}^k,\lambda^k} \quad r - \left((1+\gamma_2)\hat{\Sigma} - \hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}^{\mathsf{T}}\right) \bullet \mathbf{Q} - \hat{\Sigma} \bullet \mathbf{P} - 2\hat{\boldsymbol{\mu}}^T \mathbf{p} - \gamma_1 s$$

$$\text{s.t.} \quad \begin{bmatrix} \mathbf{P} & \mathbf{p} \\ \mathbf{p}^T & s \end{bmatrix} \succeq 0\,, \quad \mathbf{Q} \succeq 0$$

$$\begin{bmatrix} \mathbf{Q} & (\alpha_k(\mathbf{x}+\mathbf{z}^k) + 2\mathbf{p} - 2\mathbf{Q}\hat{\boldsymbol{\mu}} + \mathbf{A}^T\lambda)/2 \\ \dots & -\alpha_k\bar{\xi}^T\mathbf{z}^k + \beta_k - \mathbf{b}^T\lambda^k - r] \end{bmatrix} \succeq 0\,,\ \forall\, k$$

$$\sum_i x_i + y_i \le B$$

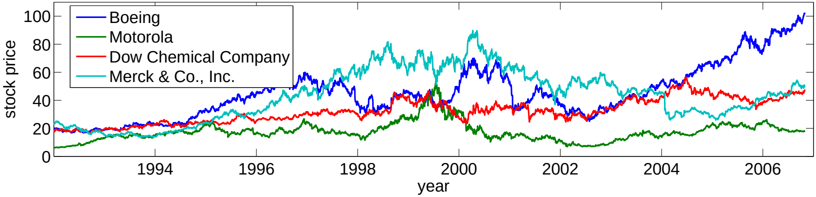
$$0 \le \mathbf{z}^k \le \mathbf{y}\,,\ \forall\, k$$

$$\lambda^k \ge 0\,, \forall\, k$$

$$\mathbf{x} \ge 0\,, \quad \mathbf{y} \ge 0$$

Experiments in Portfolio Optimization

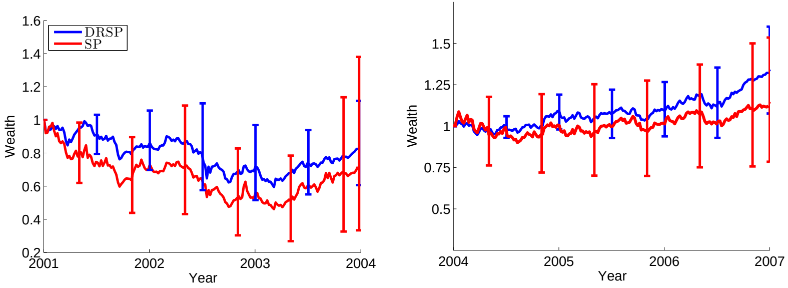
30 stocks tracked over years 1992-2007 using Yahoo! Finance



How does the robust solution perform compared to stochastic programming solution?

$$\mathcal{D} = \mathcal{D}_2(\gamma) \qquad \text{vs.} \qquad \mathcal{D} = \{\hat{F}\}$$

Wealth Evolution for 300 Experiments



- 10% and 90% percentiles are indicated periodically.

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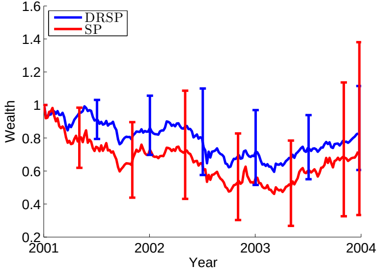

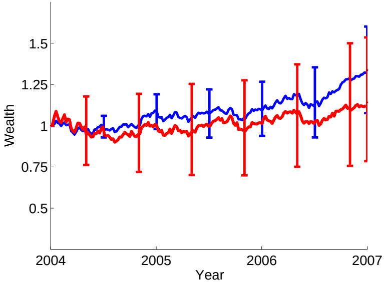

- 10% and 90% percentiles are indicated periodically.
- 79% of time, the DRSP outperformed the stoch. prog. model
- 67% improvement on average using DRSP with $\mathcal{D}_2(\gamma)$

Outline

Are Airlines Adventurous in their Fleet Acquisition?

- Fleet composition is a difficult decision problem:
  - Fleet contracts are signed 10 to 20 years ahead of schedule.
  - Many factors are still unknown at that time:
    e.g., passenger demand, fuel prices, etc.

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  - Many factors are still unknown at that time:
    e.g., passenger demand, fuel prices, etc.
- Yet, most airline companies sign these contracts based on a single scenario of what the future may be.
- Are airlines companies at risk of not being profitable in long run ?

Mathematical Formulation for Fleet Mix Optimization

The fleet composition problem is a stochastic mixed integer LP

$$\underset{\mathbf{x}}{\text{maximize}} \quad \mathbb{E}\,[-\underbrace{\mathbf{o}^\top\mathbf{x}}_{\text{ownership cost}} + \underbrace{h(\mathbf{x}, \tilde{\mathbf{p}}, \tilde{\mathbf{c}}, \tilde{\mathbf{L}})}_{\text{future profits}}\,] \;,$$

Fleet mix → $\mathbf{x}$

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$$\underset{\mathbf{x}}{\text{maximize}} \quad \mathbb{E}[-\underbrace{\mathbf{o}^T\mathbf{x}}_{\text{ownership cost}} + \underbrace{h(\mathbf{x}, \tilde{\mathbf{p}}, \tilde{\mathbf{c}}, \tilde{\mathbf{L}})}_{\text{future profits}}] \;,$$

Fleet mix → $\mathbf{x}$

with $h(\mathbf{x}, \tilde{\mathbf{p}}, \tilde{\mathbf{c}}, \tilde{\mathbf{L}}) :=$

$$\max_{z\geq 0, y\geq 0, w} \quad \sum_k \Big(\sum_i \overbrace{\tilde{p}_i^k w_i^k}^{\text{flight profit}} - \overbrace{\tilde{c}_k (z_k - x_k)^+}^{\text{rental cost}} + \overbrace{\tilde{L}_k (x_k - z_k)^+}^{\text{lease revenue}} \Big)$$

$$\text{s.t.} \quad w_i^k \in \{0,1\}\,,\ \forall\, k, \forall\, i \quad \& \quad \sum_k w_i^k = 1\,,\ \forall\, i \quad \Big\}\ \text{Cover}$$

$$y^k_{g\in\text{in}(v)} + \sum_{i\in\text{arr}(v)} w_i^k = y^k_{g\in\text{out}(v)} + \sum_{i\in\text{dep}(v)} w_i^k\,,\ \forall\, k, \forall\, v \quad \Big\}\ \text{Balance}$$

$$z_k = \sum_{v\in\{v|\text{time}(v)=0\}} \big(y^k_{g\in\text{in}(v)} + \sum_{i\in\text{arr}(v)} w_i^k\big)\,, \quad \forall k \quad \Big\}\ \text{Count}$$

The Robust Mean Value Problem

Theorem

Given any set $\mathcal{D}(\psi)$, the solution of

$$(RMVP) \quad \underset{\mathbf{x}\in\mathcal{X}}{\text{maximize}} \quad \inf_{F\in\mathcal{D}(\psi)} h(\mathbf{x}, \mathbb{E}_F[\boldsymbol{\xi}])$$

is optimal with respect to

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Proof: Since $\mathrm{h}(\mathbf{x}, \cdot)$ is convex, any feasible distribution $F \in \mathcal{D}(\psi)$ is more optimistic than the more concentrated distribution $\delta_{\mathbb{E}_F[\boldsymbol{\xi}]} \in \mathcal{D}(\psi)$:

$$\mathbb{E}_F[\mathrm{h}(\mathbf{x}, \boldsymbol{\xi})] \geq \mathrm{h}(\mathbf{x}, \mathbb{E}_F[\boldsymbol{\xi}]) = \mathbb{E}_{\delta_\mu}[\mathrm{h}(\mathbf{x}, \boldsymbol{\xi})]$$

Insights

- Resolving the mean value is more important than resolving the distribution
- The robustness of the mean value problem holds for any two-stage stochastic linear programming problem with uncertainty in objective and a risk neutral attitude
- There is a need for a tool to estimate the value of stochastic modeling

Experiments in Fleet Mix Optimization

We experimented with three test cases :

1. 3 types of aircrafts, 84 flights
2. 4 types of aircrafts, 240 flights
3. 13 types of aircrafts, 535 flights

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Test cases	Solution time		Regret for RMVP based on SP	Worst-case regret for RMVP solution
	RMVP	SP (100 scen.)		
#1	0.5 s	3 min	0.02%	$\leq$ 6%
#2	1 s	20 min	0.002%	$\leq$ 1%
#3	30 s	6 h	0.003%	$\leq$ 8%

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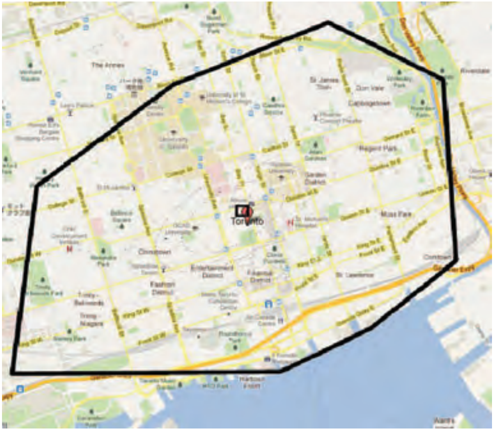
Finding:

- It would wasteful for these airline companies to invest more than 8% of profits in development of a stochastic model

Outline

Parcel delivery in Toronto

- Divide downtown Toronto region into K subregions, each serviced by a different vehicle, so that the total workload be most evenly distributed among the fleet



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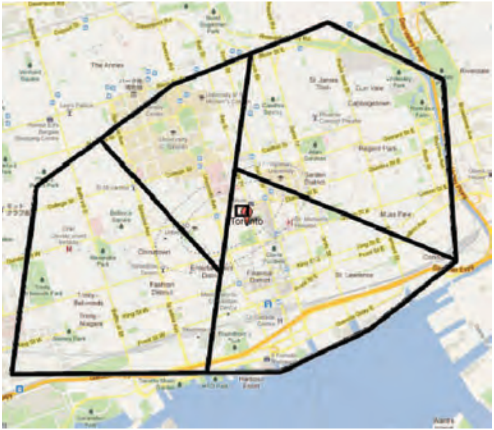

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Partitioning under a Known Demand Distribution

- Given a distribution F of demand points, we would like to partition so that the workload of the busiest driver is as small as possible

$$\min_{\{\mathcal{R}_1,\mathcal{R}_2,\ldots,\mathcal{R}_K\}} \left\{ \max_i \mathbb{E}[TSP(\{\xi_1,\xi_2,\ldots,\xi_N\}\cap\mathcal{R}_i)] \right\} \quad ,$$

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- What can we do if we only have historical samples ?

Distributionally Robust Partitioning

- Given $\mathcal{D}_2(\gamma)$, we partition so that the largest workload over the worst distribution of demand points is as small as possible

$$\min_{\{\mathcal{R}_1,\mathcal{R}_2,\ldots,\mathcal{R}_K\}} \sup_{F\in\mathcal{D}_2(\gamma)} \left\{ \max_i \iint_{\mathcal{R}_i} \sqrt{f(\xi)}d\xi \right\} \quad ,$$

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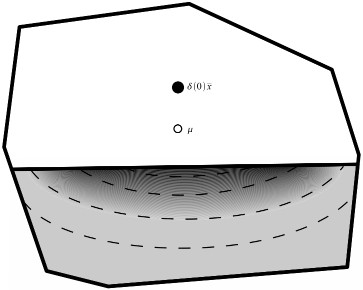
- A side product will be to characterize for any partition what is a worst-case distribution of demand locations



Evaluating the Worst-case Performance

- For a given partition, one needs to evaluate the worst-case load for each region i:

$$\max_{F \in \mathcal{D}_2(\gamma)} \iint_{\mathcal{R}_i} \sqrt{f(\xi)} d\xi$$

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- Using duality theory, we can show that it is equivalent to:

$$\begin{aligned}
\min_{r,\mathbf{Q},\mathbf{P},\mathbf{p},t,\lambda}. \quad & 1/4 \iint_{\mathcal{R}_i} \frac{1}{r + \boldsymbol{\xi}^T(2\mathbf{p} - 2\mathbf{Q}\hat{\boldsymbol{\mu}}) + \boldsymbol{\xi}^T\mathbf{Q}\boldsymbol{\xi}} d\boldsymbol{\xi} \\
& + r + ((1+\gamma_2)\hat{\Sigma} - \hat{\mu}\hat{\mu}^T) \bullet \mathbf{Q} + \hat{\Sigma} \bullet \mathbf{P} + 2\hat{\boldsymbol{\mu}}^T\mathbf{p} + \gamma_1 s \\
\text{s.t.} \quad & \begin{bmatrix} \mathbf{P} & \mathbf{p} \\ \mathbf{p}^T & s \end{bmatrix} \succeq 0 \\
& \begin{bmatrix} \mathbf{Q} & (2\mathbf{p} - 2\mathbf{Q}\hat{\boldsymbol{\mu}} + \mathbf{A}^T\lambda)/2 \\ \cdots & r - \mathbf{b}^T\lambda \end{bmatrix} \succeq 0 \qquad \lambda \geq 0 \ ,
\end{aligned}$$

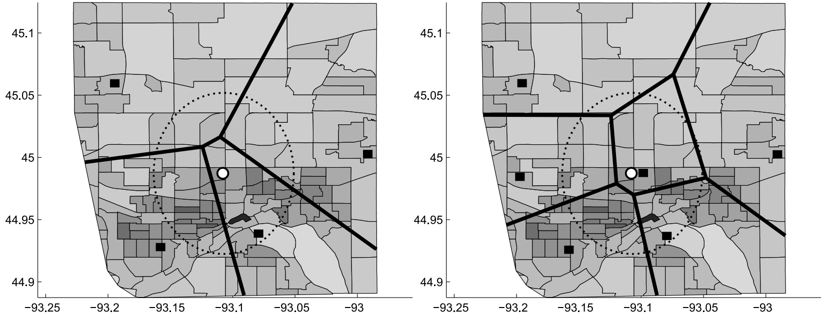
where $\mathbf{A}\boldsymbol{\xi} \leq \mathbf{b}$ describes the polygon $\mathcal{T}$.

- This problem can be formulated as an SDP after approximation of the integral by a finite sum.

Ramsey County Partition for FedEx Deliveries

Assigning each described region to a FedEx vehicle minimizes the worst-case tour of any one of them in a single day.

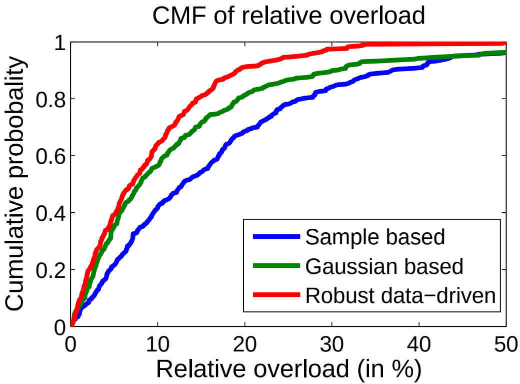
Optimal power diagrams based partitions



n=4 n=6

Distributionally Robust 2-Partitioning

We simulated three partition schemes on a set of randomly generated parcel delivery problems where the territory needed to be divided into two regions and the demand is drawn from a mixture of truncated Gaussian distribution



Outline

Conclusion

- DRSP with $\mathcal{D}(\psi)$, which imposes concentration properties, can be reformulated as a standard finite dimensional robust optimization problem
- There are cases where the robust mean value problem generates distributionally robust decisions
- DRO approach can easily lead to better decisions (both in terms of lower risk and higher returns) than an approach based on inaccurate stochastic model
- Some tools can help estimate how much identifying the true distribution is worth

Future Work

- Can a DRSP that also accounts for higher order moments be made tractable (polynomial optimization)? Would it help?

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- How to deal with ambiguity about one's risk attitude ?

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where $\mathcal{U}$ is the set of all risk averse utility functions that are coherent with a finite set of known preferences

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  where $\mathcal{U}$ is the set of all risk averse utility functions that are coherent with a finite set of known preferences
- Can lesson's learned from DRO be usefull for multi-objective problems ?

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Questions & Comments ...

... Thank you!