Accounting for Risk Measure Ambiguity when Optimizing Financial Positions

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INTRODUCTION

- Since last financial crisis, there are no more “best practice” measures of risk
  - Variance assumes symmetric distribution and considers all random variables as risky, even positive ones
  - VaR ignores what happens in the tail of distributions and does not encourage diversification of risks

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- While the axioms proposed by Artzner et al. (1999) provide good guidance, it is never easy to choose which specific measure to use
- We propose a framework for accounting precisely for what is known of the risk preferences of a decision maker when optimizing financial positions

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RISK MINIMIZATION

$Z = \xi^{\mathsf{T}} x$: the return of a financial portfolio composed by a wealth allocation vector x and securities with risky returns ξ.

The goal is to solve

$$\begin{aligned} \underset{x \in \mathcal{X}}{\text{minimize}} \quad & \rho(Z(x,\xi)) \\ s.t. \quad & \mathbb{E}\left[Z(x,\xi)\right] \geq \bar{r} \ , \end{aligned}$$

where $\rho(\cdot)$ is a risk measure and $\rho(Z_1) \geq \rho(Z_2)$ means portfolio Z_1 is perceived at least as risky as Z_2.

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Popular instances:

- **Variance**: $\rho(Z) = \mathbb{E}\left[(Z - \mathbb{E}[Z])^2\right]$: Markowitz (1952)
- **CVaR** : $\rho(Z) = -1 \cdot \mathbb{E}\left[Z | Z \leq Z_{10\%}\right]$: Rockafellar and Uryasev (2000)

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Which one and why?

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AXIOMS OF RISK MEASURES

Convex risk measures (based on diversification principle) satisfy:

1. **Monotonicity** : if $X_1 \geq X_2$ then $\rho(X_1) \leq \rho(X_2)$;
2. **Translation Invariance** : if $c \in \Re$, then $\rho(X_1 + c) = \rho(X_1) - c$,
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6. **What else ???**

... We can ask or observe the DM.

PREFERENCE ELICITATION

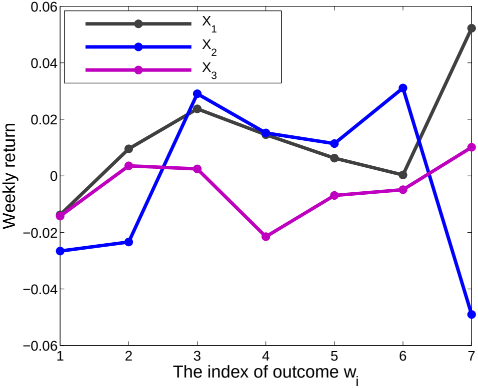
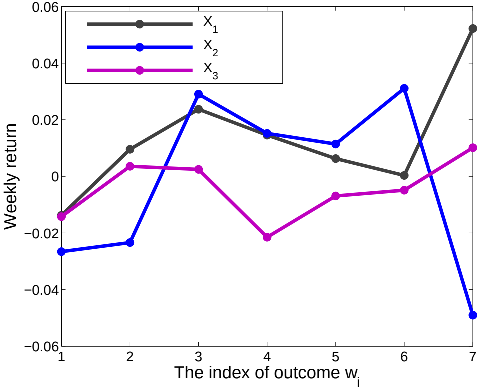


Figure: Three risk profiles

Preference Elicitation



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1. **Q1** : $X_1 ? \succeq X_3$

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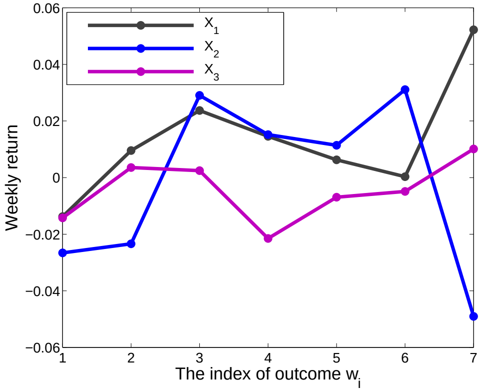

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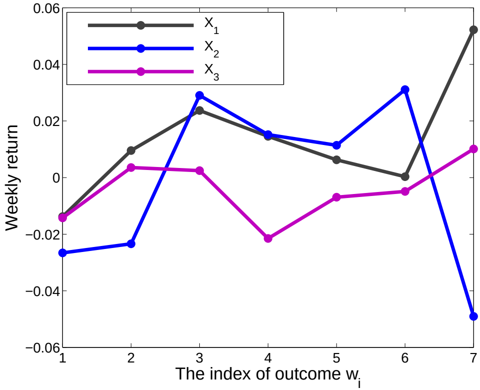

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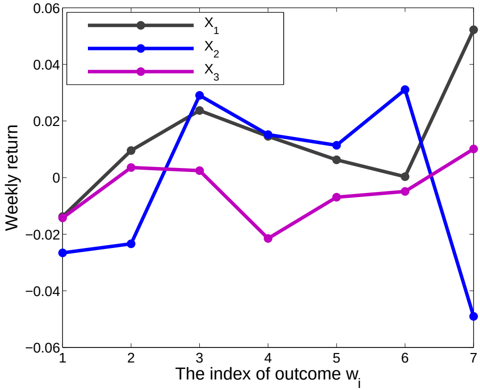

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1. **Q1** : $X_1? \succeq X_3$ **A1** : Monotonicity $\Rightarrow \rho(X_1) \geq \rho(X_3)$
2. **Q2** : $X_1? \succeq X_2$ **A2** :??(we can learn from the DM)

WHAT WE KNOW ABOUT ρ

1. **Monotonicity**
2. **Translation Invariance**
3. **Convexity**
4. **Positive homogeneity**
5. **Law invariance**
6. **Elicitation Results** : $\{\rho(X_j) \leq \rho(X_k)\}_{(j,k)\in\mathcal{I}}$

Back to the problem :

$$\min_{x\in\mathcal{X}} \rho(Z(x,\xi))$$

Question : How should we choose the portfolio x when only the above information about ρ is known ?

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Robust Optimization Formulation

We propose the following minmax formulation

$$\min_{x \in \mathcal{X}} \sup_{\rho \in \mathcal{R}} \rho(Z(x, \xi)),$$

where $\mathcal{R} := \{\rho \mid \rho$ satisfies a subset of (1) to (6)$\}$

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THE ROBUST MEASURE AND OPTIMIZATION

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Fact 1 :
If ρ is a convex/coherent/law inv. risk measure, then
$\rho' = \sup_{\rho \in \mathcal{R}} \rho$ is also a convex/coherent/law inv. risk measure.

The Robust Measure and Optimization

Fact 1 :
If ρ is a convex/coherent/law inv. risk measure, then
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Fact 2 :
Assuming that

- Set $\mathcal{X}$ is convex
- Random vector ξ has N possible outcomes

then the risk vs. return optimization problem is a convex optimization problem that can be solved in polynomial time.

Robust Convex Risk Measure

Let's consider the problem

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad \sup_{\rho \in \mathcal{R}_1} \rho(Z(x, \xi)) \ ,$$

where $\mathcal{R}_1 := \{\rho \mid \rho \text{ satisfies conditions (1), (2), (3), and (6)}\}$

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ROBUST CONVEX RISK MEASURE

The optimization problem can be equivalently formulated as the following finite dimensional convex optimization problem

$$(P) \quad \min_{x \in \mathcal{X}, t, \theta} \quad t$$
$$\text{s.t.} \quad Z(x, \xi_i) + t \geq [X_1(\xi_i) \cdots X_m(\xi_i)]\theta + {\delta^*}^\top \theta, \ \forall i = 1, \ldots, N$$
$$\mathbf{1}^\top \theta = 1, \ \theta \geq 0 \ ,$$

where δ^* is the optimal solution of

$$\max_{\delta, \lambda} \quad \sum_i \delta_i$$
$$\text{s.t.} \quad \delta_j \leq \delta_k, \ \forall (j, k) \in \mathcal{I}$$
$$\delta_j \geq \delta_i - \lambda_i^\top (X_j - X_i), \ \forall i, j$$
$$\mathbf{1}^T \lambda = 1 \ \& \ \lambda_i \geq 0, \ \forall i$$

OUTLINE OF THE REFORMULATION PROCEDURE

We reformulate the problem from the perspective of acceptance sets (Föllmer and Schied 2002)

$$\rho_{\mathcal{A}}(Z) := \inf_{t \in \Re} \{t \mid Z + t \in \mathcal{A}\}.$$

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Our goal is to characterize the **worst-case** set $\mathcal{A}^*$ for a risk profile Z

$$\sup_{\mathcal{A} \in \mathbb{A}} \rho_{\mathcal{A}}(Z),$$

where $\mathbb{A} := \left\{ \mathcal{A} \middle| \begin{array}{l} \rho_{\mathcal{A}} = \text{convex risk measure} \\ \rho_{\mathcal{A}}(X_j) \leq \rho_{\mathcal{A}}(X_k), \ \forall (j,k) \in \mathcal{I} \end{array} \right\}$ denotes **a set of acceptance sets** that are consistent with given information.

Outline of the Reformulation Procedure

We prove that the worst-case measure satisfies for all Z:

$$\sup_{\mathcal{A}\in\mathbb{A}} \rho_{\mathcal{A}}(Z) \quad = \quad \sup_{\delta\in\Delta} \sup_{\mathcal{A}\in\mathbb{A}(\delta)} \rho_{\mathcal{A}}(Z)$$

where

$$\mathbb{A}(\delta) := \left\{ \mathcal{A} \middle| \begin{array}{l} \rho_{\mathcal{A}} = \text{convex risk measure} \\ \rho_{\mathcal{A}}(X_i) = \delta_i, \ \forall\, i \end{array} \right\}$$

and

$$\Delta = \{\delta \in \Re^m \mid \mathbb{A}(\delta) \neq \emptyset \ \&\ \delta_j \leq \delta_k, \ \forall\, (j,k) \in \mathcal{I}\}$$

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and $\mathcal{H}(\delta)$ is a convex polyhedron with the points $\{X_i + \delta_i\}$ as vertices

$$\mathcal{H}(\delta) = \left\{ y \in \Re^N \,\middle|\, \exists\theta \in \Re^m,\ \begin{array}{l} y \geq [X_1\ X_2\ \cdots X_m]\theta + \delta^T\theta \\ \mathbf{1}^\top\theta = 1\ ,\ \ \theta \geq 0 \end{array} \right\}$$

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DETAILS OF EXPERIMENTS

We consider a static portfolio optimization problem with 4 assets over a period of one week

$$\min_{x \geq 0, \mathbf{1}^\top x = 1} \rho(R^\top x)$$

We simulate a decision maker's true risk attitude using the following unknown law inv. coherent risk measure

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- On any given week, last 13 weeks' returns constitute the outcome space
- For elicitation, we use a number of 13 weeks risk profiles from 2007 and 2008
- We report on 4000 experiments. For each one:
  - We randomly draw a date between 2009 and 2012
  - We randomly draw 4 assets for portfolio optimization
  - Performance of obtained portfolios is measured using true risk measure

NUMERICAL RESULTS

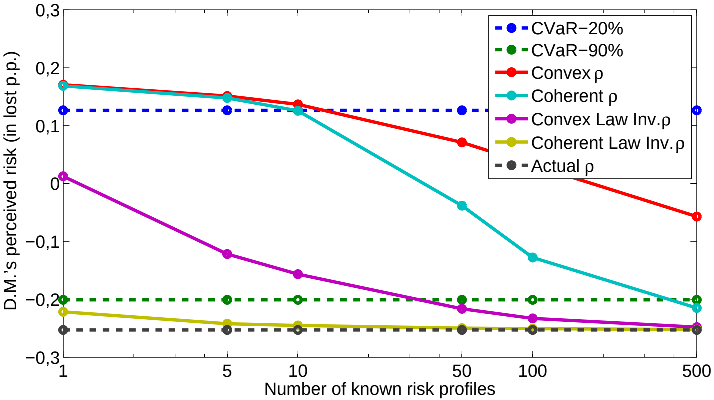

Figure: Average perceived risk for the optimized portfolios over 4000 experiments

CONCLUSION

- Assuming a particular form of CVaR can be misleading, one can instead use an ambiguity averse risk measure formulation
- Impact of information about global attitude is significant but can be replaced with information about risk profiles
- The measures that account for law invariance can achieve nearly optimal performance with a small amount of additional information

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- In Armbruster and Delage (2012), we develop a similar framework but for expected utility theory